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Canonical Localizers and Nonmaximal Orders in the Witt Setting

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Canonical localizers relating the number field trace to the residue field trace arise in the Witt group of degree k maps and in the study of the knot concordance group. These are required to do computations in the Witt setting when nonmaximal orders in an algebraic number field are studied. This paper gives criteria under which these canonical localizers can be computed. It also gives an example of an element of order 4 in the knot concordance group $C_S(Z)$, where $S = Z(\theta)$ is a nonmaximal order in the algebraic number field $Q(\theta)$.

1. The Setting and Problem

Consider a triple (M, B, f) satisfying:

(1) M is a finitely generated torsion free Z-module.

(2) $B: M \times M \to Z$ is a symmetric Z-valued inner product defined on M.

(3) $f: M \to M$ is a map of degree k, meaning $B(fx, fy) = kB(x, y)$ for all x, y in M.

After placing the Witt equivalence relation on these triples [7] we obtain the Witt group $W(k, Z)$.

When $k = +1$ this is precisely the Witt group arising in Kreck's work [2] on determining the bordism group of orientation preserving diffeomorphisms of closed smooth oriented n-dimensional manifolds. The generalization to degree k maps has algebraic implications in the exact octagon derived in [7].

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If we replace condition (3) by

(3′) $f: M \to M$ satisfies $B(fx, y) = B(x, (1-f)y)$

there results the knot concordance group discussed by Stoltzfus [6].

Next place the additional requirement

(4) f satisfies the monic integral irreducible polynomial $p(t)$

on the map f of the degree k mapping structure (M, B, f). The action of f induces a $z[t]/(p(t))$-module structure on M. To simplify our notation we write $S = Z[t]/(p(t))$. Observe that S is only an order in the algebraic number field $Q[t]/(p(t)) = E$, and may not be the maximal order $D = O(E)$, which is the Dedekind ring of integers in E. The resulting Witt group of triples satisfying conditions (1)–(4) forms a subgroup of $W(k, Z)$ and is denoted by $W(k, Z; p)$ or $W(k, Z; S)$. Similarly there is the knot concordance group $C_p(Z) = C_S(Z)$.

The fact that f is a map of degree k will yield an involution denoted by $-$ on E. This $-$ involution is given by $t \to kt^{-1}$ and $t^{-1} \to k^{-1}t$. For the knot concordance group the involution is $t \to 1 - t$. Let F denote the fixed field of $-$. The Dedekind rings of integers in E and F are denoted by $O(E)$ and $O(F)$, respectively. We will also write $O(E) = D$.

If $\mathscr{P}$ is a prime ideal in $O(E)$, then $P = \mathscr{P} \cap O(F)$ is the corresponding prime ideal in $O(F)$. If I is a fractional $O(E)$-ideal, then $\operatorname{ord}_{\mathscr{P}} I$ is the exponent to which $\mathscr{P}$ is raised in the factorization of I.

The computation of the relative Witt groups $W(Z; S)$ of integral forms with compatible S-module structure has been made in [6] and [7] using the Knebusch localization sequence and the identification of $W(Z; D)$ with $H(\Delta^{-1}(D/Z))$ via the trace form, where D is the maximal order and $\Delta^{-1}(D/Z)$ is the inverse different. A crucial component of this computation is that of the boundary

$$\begin{tikzcd}
H(E) \arrow[r, "\partial(D)"] \arrow[d, "\mathrm{tr}"] & H(E/\Delta^{-1}(D/Z)) \arrow[r] \arrow[d, "\mathrm{tr}"] & H(D/\mathscr{P}) \arrow[d, "\mathrm{Tr}"] \\
W(k, Q; D) \arrow[r, "\partial(D)"] & W(k, Q/Z; D) \arrow[r] & W(k, Q/Z; D/\mathscr{P}),
\end{tikzcd}$$

where tr is induced by the number field trace and Tr is induced by the finite field trace.

At every prime $\mathscr{P}$ in D there exists a canonically defined element $\rho_{\mathscr{P}}$ in $E/\Delta^{-1}(D/Z)$ with the following properties. This element $\rho_{\mathscr{P}}$ has order one less than the inverse different $\Delta^{-1}(D/Z)$, i.e., $\operatorname{ord}_{\mathscr{P}}(\rho_{\mathscr{P}}) = \operatorname{ord}_{\mathscr{P}}(\Delta^{-1}(D/Z)) - 1$. The map of $O(E) \to E/\Delta^{-1}(D/Z)$ given by $x \to x\rho_{\mathscr{P}}$ induces an embedding of the residue field $O(E)/\mathscr{P} \to E/\Delta^{-1}(D/Z)$. The map of $Z \to Q/Z$ given by $n \to n(1/p)$ induces an embedding of $Z/pZ = F_p \to Q/Z$.

The element $\rho_{\mathscr{P}}$ is canonical in the sense that the following diagram commutes:

$$\begin{array}{ccc} O(E)/\mathscr{P} & \longrightarrow & E/\Delta^{-1}(D/Z) \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{tr}} \\ F_p & \longrightarrow & Q/Z \end{array}$$

DIAGRAM 1

The horizontal maps were just described. Tr again denotes the map induced by the finite field trace; tr denotes the map induced by the number field trace. This construction extends results of [1] and [7] and will be given in Section 2 along with several examples of the computation of canonical localizers.

In the case when the order S is nonmaximal the computation of the boundary involves both:

(1) the trace map $\mathrm{Tr}: H(D/\mathscr{P}) \to H(S/\mathscr{P} \cap S)$,

(2) $T(\mathscr{M}) = \{\mathscr{P}: \mathscr{P}$ is a $-$ invariant maximal ideal in D satisfying $\mathscr{P} \cap S = \mathscr{M}\}$.

The trace map Tr on finite fields is computed in [6, p. 34] and extends results of Milnor [4]. $T(\mathscr{M})$ is more complicated and is related to the conductor of S in the maximal order D. If $\mathscr{M}$ is prime to C, its cardinality $\#T(\mathscr{M})$ is one, but the cardinality can be greater than one otherwise. In Section 3 we construct an example of an element of order four in $C_S(Z)$, contradicting Corollary 5.4 of [6]. This phenomena may occur, however, only when S is nonmaximal and $\#T(\mathscr{M})$ is greater than 1 for some $\mathscr{M}$, a possibility which was overlooked in [6]. Another unpublished counterexample in $Q(\sqrt{3}, \sqrt{7})$ with minimal polynomial $p(x) = x^2(1-x)^2 + 148x(1-x) + 373$ has been found by N. W. Stoltzfus. An explicit Seifert form may be constructed by Levine's method for realization of Alexander polynomials.

2. Canonical Localizers

We begin this section by reviewing precisely where the canonical localizers $\rho_{\mathscr{P}}$ come from. The reader is referred to [1] or [7] for a more complete exposition. We then show how in certain instances to compute them. To begin with work over a local complete field. There is no loss of generality in doing this as we shall see in Theorem 2.8.

Consider the extension E/K of the complete field K containing $\tilde{Q}_p$ the p-adic rationals. For any prime ideal $\mathscr{P} \subset O(E)$ lying over $P \subset O(K)$ we have the diagram

$$\begin{array}{ccc} O(E) & \xrightarrow{v_{\mathscr{P}}} & O(E)/\mathscr{P} = E_{\mathscr{P}} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Tr}_{E_{\mathscr{P}}/K_P}} \\ O(K) & \xrightarrow{v_P} & O(K)/P = K_P. \end{array}$$

The maps $v_{\mathscr{P}}$ and v_P are projections onto the finite residue fields. Since $O(E)$ is a finitely generated projective $O(K)$-module (free since $O(K)$ is a principal ideal domain), we can find an $O(K)$-module homomorphism $f\colon O(E) \to O(K)$ satisfying

$$v_P \circ f = \mathrm{Tr}_{E_{\mathscr{P}}/K_P} \circ v_{\mathscr{P}}. \tag{1}$$

Further we identify $\Delta^{-1}(E/K) = \Delta^{-1}$ with $\mathrm{Hom}_{O(K)}(O(E), O(K))$ by $\mu \to \mathrm{tr}_{E/K}(\mu -)$. It follows that we can find $\mu_{E/K} = \mu \in \Delta^{-1}$ such that $f(x) = \mathrm{tr}_{E/K}(\mu x)$ for all $x \in O(E)$. Thus we have

$$v_P(\mathrm{tr}_{E/K}(\mu x)) = \mathrm{Tr}_{E_{\mathscr{P}}/K_P}(v_{\mathscr{P}}(x)) \qquad \text{for all } x \in O(E). \tag{2}$$

Conner has developed the basic properties of $\mu_{E/K}$ which we record in

LEMMA 2.1. (1) *μ is unique modulo $P\Delta^{-1} \subset \Delta^{-1}$,*

(2) $\mu \notin P\Delta^{-1}$,

(3) $\mu\mathscr{P} \subset P\Delta^{-1}$,

(4) $\mathrm{ord}_{\mathscr{P}}\mu = \mathrm{ord}_{\mathscr{P}} P + \mathrm{ord}_{\mathscr{P}} \Delta^{-1} - 1$.

Proof. See [7, p. 187]. ∎

Of course $\mathrm{ord}_{\mathscr{P}} P = e$ the ramification index of E/K.

Next let $\pi \in O(K)$ be a local uniformizer. Set $\rho_{E/K} = \rho = \mu/\pi$, so $\mathrm{ord}_{\mathscr{P}} \rho = \mathrm{ord}_{\mathscr{P}} \Delta^{-1} - 1$; ρ is called a localizer for E/Δ^{-1}. We then obtain an embedding of the residue field $O(E)/\mathscr{P} = E_{\mathscr{P}}$ given by

$$E_{\mathscr{P}} \to E/\Delta^{-1} \qquad \text{given by } x \to (x\rho) \in E/\Delta^{-1} \qquad \text{for all } x \in O(E).$$

Note that $(\rho) \in E/\Delta^{-1}$ and hence the embedding depends only on π and not on μ by Lemma 2.1(1).

We embed the residue field $O(K)/P = K_P \to K/(O(K)$ by $x \to (x/\pi)$. The definitions are made precisely so that we obtain the commutative diagram relating the number field trace $\mathrm{tr}_{E/K}$ to the residue field trace $\mathrm{Tr}_{E_{\mathscr{P}}/K_P}$:

$$\begin{array}{ccccc} O(E)/\mathscr{P} = E_{\mathscr{P}} & \longrightarrow & E/\Delta^{-1} & & x \to (x\mu/\pi) \\ {\scriptstyle \mathrm{Tr}_{E_{\mathscr{P}}/K_P}} \downarrow & & \downarrow {\scriptstyle \mathrm{tr}_{E/K}} & & \\ O(K)/P = K_P & \longrightarrow & K/O(K) & & x \to x/\pi. \end{array}$$

DIAGRAM 2

If we choose $K = \tilde{Q}_p$ the p-adics, and let $\pi = p$, then it follows that Diagram 2 can be viewed as the "completion" of Diagram 1. Thus we see where the canonical localizers $\rho_{\mathscr{P}}$ determining the isomorphism $H(E/\Delta^{-1}(E/Q)) \simeq \oplus_{\mathscr{P} = \bar{\mathscr{P}}} H(E_{\mathscr{P}})$ come from. We must use these canonical localizers to have commutative diagrams for computing [7, p. 148]. To compute the local boundary $\partial_{\mathscr{P}}: H(E) \to W(F_p)$ at nondyadic ramified primes, Section 3, Theorem 3.1, we shall need to know whether $\rho_{\mathscr{P}} = \mu/p$ is a local norm.

Again, for now we are assuming E is complete. Let us make some general remarks about μ and hence about $\rho_{\mathscr{P}} = \mu/\pi$ which can be used later on. By Lemma 2.1 there is a unique coset $(\mu_{E/K}) \in \Delta^{-1}/P\Delta^{-1}$, where $\Delta^{-1} = \Delta^{-1}(E/K)$ is the inverse different, and we have the following facts about μ:

LEMMA 2.2. *If μ represents $(\mu_{E/K})$, then so does μV, where $V \in O(E)^*$ is any unit with $v_{\mathscr{P}}(V) = 1$.*

Proof. Since $v_{\mathscr{P}}(V) = 1$, $V = 1 + z$ for some $z \in \mathscr{P}$. Thus $\mu - \mu V = \mu(-z) \in P\Delta^{-1}$ by Lemma 2.1(3). Hence $(\mu) = (\mu V)$. ∎

LEMMA 2.3. *If both μ and μ_1 represent the coset $(\mu_{E/K})$ in $\Delta_{E/K}^{-1}/P\Delta_{E/K}^{-1}$, then there exists a unit $V \in O(E)^*$ with $v_{\mathscr{P}}(V) = 1$ and $\mu_1 = \mu V$.*

Proof. By 2.1, $\mathrm{ord}_{\mathscr{P}} \mu = \mathrm{ord}_{\mathscr{P}} \mu_1$, so the quotient $\mu_1/\mu = V$ a unit in $O(E)^*$. Now $\mu - \mu_1 = \mu - \mu V = \mu(1 - V) \in P\Delta^{-1}$, but $\mu \notin P\Delta^{-1}$. Thus $\mathrm{ord}_{\mathscr{P}}(1 - V) \geqslant 1$, and $1 - V \in \mathscr{P}$. Thus $v_{\mathscr{P}}(V) = 1$. ∎

LEMMA 2.4. *If E/K is unramified then* 1 *represents $\mu_{E/K}$.*

Proof. Clear from the defining equation (2) for $\mu_{E/K}$. ∎

LEMMA 2.5. *For $K \subset F \subset E$, choose representatives $\mu_{F/K} \in \Delta_{F/K}^{-1}$ and $\mu_{E/F} \in \Delta_{E/F}^{-1}$. Then the product $\mu_{E/F} \cdot \mu_{F/K} \in \Delta_{E/K}^{-1}$ represents $\mu_{E/K}$.*

Proof. See [7, p. 196]. ∎

We now show how to compute $\mu_{E/K}$ in several examples. Let E/K be a finite extension, with e the ramification index and $f=\deg E_{\mathscr{P}}/K_P$ the residue field degree.

THEOREM 2.6. *If* $\operatorname{ord}_{\mathscr{P}} \Delta_{E/K}^{-1} \equiv 1 \pmod{e}$ *then* $1/e$ *represents* $(\mu_{E/K})$.

Proof. We begin by splitting E/K into $K \subset F \subset E$, with F/K unramified of degree f and E/F totally ramified of degree e. Let $\pi \in O(K)$ be a local uniformizer for $O(K)$ and $O(F)$. Let μ represent $\mu_{E/F}$. Then by Lemma 2.1,

$$\operatorname{ord}_{\mathscr{P}} \mu = e + \operatorname{ord}_{\mathscr{P}} \Delta_{E/F}^{-1} - 1. \tag{3}$$

We assumed $\operatorname{ord}_{\mathscr{P}} \Delta_{E/F}^{-1} - 1 \equiv 0 \pmod{e}$. Hence we may write $\mu = \pi^m U$ for some integer m, and some unit $U \in O(E)^*$. Note that $\pi \in O(F)$. We let $\mathscr{P} \in O(E)$ lie over $\mathfrak{p} \in O(F)$ which lies over $P \in O(K)$.

Since E/F is totally ramified, the residue fields $E_{\mathscr{P}}$ and $E_{\mathfrak{p}}$ are equal, $E_{\mathscr{P}} = F_{\mathfrak{p}}$. Thus we can find $V \in O(F)^*$ with $v_{\mathscr{P}}(U) = v_{\mathfrak{p}}(V) = v_{\mathscr{P}}(V)$. Since $v_{\mathscr{P}}(VU^{-1}) = 1$ we may apply Lemma 2.2 and replace μ by $\mu VU^{-1} = \pi^m V$. Thus we obtain a representative of $(\mu_{E/F})$, still denoted by μ, which lies in the subfield F. Recall F/K is unramified so by Lemma 2.4 1 represents $\mu_{F/K}$. Then $1 \cdot \mu = \mu$ also represents $(\mu_{E/K})$ by Lemma 2.5.

Consider $e\mu$. For $x \in O(F) \subset O(E)$, $\operatorname{tr}_{E/K}(\mu x) = \operatorname{tr}_{F/K}(e\mu x)$, so that

$$\begin{aligned} v_P(\operatorname{tr}_{E/K}(\mu x)) &= \operatorname{Tr}_{E_{\mathscr{P}}/K_P}(v_{\mathscr{P}}(x)) \\ &= \operatorname{Tr}_{F_{\mathfrak{p}}/K_P}(v_{\mathfrak{p}}(x)) \\ &= v_P(\operatorname{tr}_{F/K}(e\mu x)). \end{aligned}$$

Hence $e\mu$ also represents $(\mu_{F/K})$. Thus by Lemma 2.3 there is a unit $W \in O(F)^*$ with $e\mu W = 1$ and $v_{\mathfrak{p}}(W) = v_{\mathscr{P}}(W) = 1$. Thus $\mu W = 1/e$. But again by Lemma 2.2, μW also represents $(\mu_{E/K})$. ∎

The hypothesis of Theorem 2.6 is always satisfied for tamely ramified extensions because in that case $\operatorname{ord}_{\mathscr{P}} \Delta^{-1} = 1 - e$. It is also satisfied for some wildly ramified extensions.

EXAMPLE 1. Let $E = Q(\lambda)$, where λ is a primitive p^rth root of unity. Then p is the only ramified prime, and p is totally ramified with ramification index $e = (p-1)p^{r-1}$.

Let $f(t)$ be the irreducible polynomal for λ. Then $f(1-t) = g(t)$ is the irreducible polynomial for $(1-\lambda) = \theta$ which is a local uniformizer for $\tilde{O}_E(\mathscr{P})$, the local completion at the prime $\mathscr{P}$. Note that $g(0) = f(1) = p$ the chosen local uniformizer for $\tilde{Q}_p$. Computing the derivative $g'(\theta)$, we find that $\operatorname{ord}_{\mathscr{P}} \Delta^{-1} = 1 - er \equiv 1 \pmod{e}$ (see [8, p. 266]). Thus, when $r > 1$, although the ramification is wild we may still apply Theorem 2.6 to find a localizer.

EXAMPLE 2. Let $E = Q(\theta)$, where θ satisfies the irreducible polynomial $f(t) = t^p - p$. The prime p is again totally and wildly ramified. Locally, the different is generated by $f'(\theta) = p\theta^{p-1} = \theta^{2p-1}$. Again it follows that $\operatorname{ord}_{\mathscr{P}} \Delta^{-1} = 1 - 2p \equiv 1 \pmod{e}$ since $e = p$.

EXAMPLE 3. Let $E = Q(\theta)$, where θ satisfies the irreducible polynomial $f(t) = t^{2p} + pt + p$. The prime p is totally and wildly ramified since the polynomial is Eisenstein at p. Since E is totally ramified, when we complete, we find that the different is generated by $f'(\theta) = p(2\theta^{2p-1} + 1)$. This has order $e = 2p$ at $\mathscr{P}$ lying over p. Hence Theorem 2.6 does not apply. To compute the localizer ρ we need the following.

Consider E/F totally ramified of degree e. Assume F is complete at the nondyadic prime $\mathfrak{p}$, so that the extension is given by an Eisenstein polynomial $f(t)$ of degree e over $O(F)$. Let $f(0) = \pi$ be the chosen local uniformizer for $O(F)$. The adjoined root θ is a local uniformizer for $O(E)$ and primitively generates $O(E)$ as an $O(F)$-module.

We can explicitly construct $\mu_{E/F}$ in this case as follows. We begin with $\{1, \theta, ..., \theta^{e-1}\}$ which is a basis for E/F. The dual base with respect to $\operatorname{tr}_{E/F}$ is given as follows. Let $f(t) = (t - \theta)(b_0 + b_1 t + \cdots + b_{e-1} t^{e-1})$. Then the dual base is $\{b_j/f'(\theta), j = 0 \cdots (e-1)\}$. Specifically $b_0 = -\pi/\theta$ so that $-\pi/\theta f'(\theta)$ is dual to 1. The defining equation (2) for μ reduces to $v_{\mathfrak{p}}(\operatorname{tr}_{E/F}(\mu x)) = v_{\mathscr{P}}(x)$ since $E_{\mathscr{P}} = F_{\mathfrak{p}}$ in the totally ramified case. It follows that we may choose $\mu = -\pi/\theta f'(\theta)$ to represent $(\mu_{E/F})$ since $v_{\mathscr{P}}(\theta)^j = 0 \in E$, $1 \leqslant j \leqslant e-1$, while $v_{\mathscr{P}}(1) = v_{\mathfrak{p}}(1) = 1$. This construction applies in Example 3.

We should also observe that the converse of Theorem 2.6 is true.

THEOREM 2.7. *Suppose* $\mu_{E/K} = 1/e$. *Then* $\operatorname{ord}_{\mathscr{P}} \Delta^{-1} \equiv 1 \pmod{e}$.

Proof. Apply Lemma 2.1(4) to write $\operatorname{ord}_{\mathscr{P}} \Delta^{-1} = \operatorname{ord}_{\mathscr{P}} \mu - \operatorname{ord}_{\mathscr{P}} P + 1$. Since $\mu \in K$, it follows that $\operatorname{ord}_{\mathscr{P}} \mu \equiv 0 \pmod{e}$, and $\operatorname{ord}_{\mathscr{P}} P = e \equiv 0 \pmod{e}$ also. The result follows. ∎

Again, in the above we have assumed that we could complete at the prime $\mathscr{P}$ and then work locally. We now justify this in Theorem 2.8. We work over the global field E and show that we may choose an element $\tilde{\mu}(\mathscr{P})$ to be a globally chosen element $\mu \in E$.

Let E/Q be a finite extension of the rationals. Let $O(E) \subset E$ be the Dedekind ring of integers. If $\mathscr{P} \subset O(E)$ is a prime ideal, then $\mathscr{P}$ lies over a rational prime $p \in Z$. In fact, given $p \in Z$ we form $pO(E) = \mathscr{P}_1^{e_1} \cdots \mathscr{P}_r^{e_r}$. Again $T(p) = \{\mathscr{P} \subset O(E) \colon \mathscr{P} \cap Z = (p)\}$. Then $\mathscr{P} \in T(p)$ if and only if $\operatorname{ord}_{\mathscr{P}} p \geqslant 1$.

Given $\mathscr{P}$, p as above, we have the quotient homomorphisms $v_{\mathscr{P}} \colon O(E) \to O(E)/\mathscr{P} = E_{\mathscr{P}}$ and $v_p \colon Z \to Z/pZ = F_p$ onto the finite residue fields exactly as before, except now E is not complete.

Let $\Delta_{E/Q}^{-1} = \Delta^{-1}$ be the inverse different of E/Q. As before we can find $\mu \in \Delta^{-1}$ such that $v_p(\mathrm{tr}_{E/Q}(\mu x)) = \mathrm{Tr}_{E_{\mathscr{P}}/F_p}(v_{\mathscr{P}}(x))$ for all $x \in O(E)$. Again $\mathrm{ord}_{\mathscr{P}}\mu = \mathrm{ord}_{\mathscr{P}} p + \mathrm{ord}_{\mathscr{P}} \Delta^{-1} - 1$ and Lemma 2.1 applies. In particular $\mu\mathscr{P} \subset \Delta^{-1}(p)$. Thus, if $\mathscr{P}_1 \neq \mathscr{P}$ we obtain

$$\mathrm{ord}_{\mathscr{P}_1} \mu + \mathrm{ord}_{\mathscr{P}_1} \mathscr{P} \geqslant \mathrm{ord}_{\mathscr{P}_1} \Delta^{-1} + \mathrm{ord}_{\mathscr{P}_1} p. \tag{4}$$

However, $\mathrm{ord}_{\mathscr{P}_1} \mathscr{P} = 0$ since $\mathscr{P}_1 \neq \mathscr{P}$, so we obtain

$$\mathrm{ord}_{\mathscr{P}_1} \mu \geqslant \mathrm{ord}_{\mathscr{P}_1} \Delta^{-1} + \mathrm{ord}_{\mathscr{P}_1} p. \tag{5}$$

The term μ is only unique modulo $p\Delta^{-1} \subset \Delta^{-1}$.

We will use the notation $\tilde{}$ to distinguish the completion and choose $\tilde{\mu}$ as before.

THEOREM 2.8. *We can choose the local $\tilde{\mu}(\mathscr{P})$ to be a globally chosen μ.*

Proof. For $x \in E$, $\mathrm{tr}_{E/Q}(x) = \mathrm{tr}_{\tilde{E}(\mathscr{P})/\tilde{Q}(p)}(x) + \sum_{\mathscr{P}_1 \neq \mathscr{P}} \mathrm{tr}_{\tilde{E}(\mathscr{P}_1)/\tilde{Q}(p)}(x)$, where the sum runs over all $\mathscr{P}_1$ dividing p and $\mathscr{P}_1 \neq \mathscr{P}$.

Recall $\mathrm{ord}_{\mathscr{P}_1} \Delta^{-1} = \mathrm{ord}_{\mathscr{P}_1} \tilde{\Delta}^{-1}(\mathscr{P}_1))$ by [3, p. 61]. Set $e_1 = \mathrm{ord}_{\mathscr{P}_1} p$. Then locally we have $p\tilde{\Delta}^{-1}(\mathscr{P}_1) = \mathscr{P}_1^{e_1}\tilde{\Delta}^{-1}(\mathscr{P}_1)$ and $\mu \in \mathscr{P}_1^{e_1}\tilde{\Delta}^{-1}(\mathscr{P}_1)$ by Eq. (5). Hence for $x \in O(E)$ we have $\mathrm{tr}_{\tilde{E}(\mathscr{P}_1)/\tilde{Q}(p)}(\mu x) \in p\tilde{Z}(p)$. Thus

$$v_p(\mathrm{tr}_{E/Q}(\mu x)) = v_p(\mathrm{tr}_{\tilde{E}(\mathscr{P})/\tilde{Q}(p)}(\mu x)) \tag{6}$$

for all $x \in O(E)$. Since $O(E)$ is dense in $\tilde{O}_E(\mathscr{P})$, (6) holds for all $x \in \tilde{O}_E(\mathscr{P})$. Since the residue fields $\tilde{O}_E(\mathscr{P})/\tilde{\mathscr{P}} = E_{\mathscr{P}} = O(E)/\mathscr{P}$ it follows that we could take $\tilde{\mu} = \mu$, i.e., μ can be chosen to represent $(\tilde{\mu}) \in \tilde{\Delta}^{-1}(\mathscr{P})/p\tilde{\Delta}^{-1}(\mathscr{P})$. ∎

Therefore in computing $\partial(D, \mathscr{P})$: $H(E) \to H(E_{\mathscr{P}})$ (or W_p) for $\mathscr{P}$ ramified), the completion of E may be made at $\mathscr{P}$ before computing. This is because the residue field $E_{\mathscr{P}}$ may be embedded using the same canonical localizers in either case. Hence there is no loss of generality in studying the local complete case as we have done.

3. THE SET $T(\mathscr{M})$ AND NONMAXIMAL ORDERS

To show how $T(\mathscr{M})$ affects the boundary, we first need to recall the boundary computation for $\partial(D, \mathscr{P})$. Here is how it works. If $\mathscr{P} = \bar{\mathscr{P}}$ is inert, we obtain an identification $H(E/\Delta_{E/Q}^{-1}(\mathscr{P})) \simeq H(E_{\mathscr{P}}) \simeq Z^*$ which is completely independent of the choice of localizer. If $\mathscr{P} = \bar{\mathscr{P}}$ is ramified we obtain an identification $H(E/\Delta_{E/Q}^{-1}(\mathscr{P})) \simeq H(E_{\mathscr{P}}) \simeq W(F_p)$. If $\mathscr{P} = \bar{\mathscr{P}}$ is dyadic ramified identify $W(F_{\mathfrak{p}}) \simeq Z/2Z$ by rank mod 2. If $\mathscr{P} = \bar{\mathscr{P}}$ is nondyadic ramified, identify the image of $\partial(D, \mathscr{P})$ as a ring as

$$W(F_{\mathfrak{p}}) \simeq Z^* \times Z_2 \qquad (d_1, e_1) + (d_2, e_2) = ((-1, \sigma)_{\mathfrak{p}}^{e_1 e_2} d_1 d_2, e_1 + e_2)$$
$$(d_1, e_1) \cdot (d_2, e_2) = (d_1^{e_2} d_2^{e_2}, e_1 e_2).$$

Here $E = F(\sqrt{\sigma})$ and $(-, \sigma)_{\mathfrak{p}}$ is the Hilbert symbol read at $\mathfrak{p}$. Of course the reader should view this as d_1 is the discriminant and e_1 is the rank mod 2.

THEOREM 3.1. *Under the above identifications, $\partial(D, \mathscr{P})$ is read using canonical localizers as follows:*

(1) *If $\mathscr{P} = \bar{\mathscr{P}}$ over inert then*

$$\partial(D, \mathscr{P})(V, h) = (-1)^{(rk(V,h))(\operatorname{ord}_{\mathscr{P}} \Delta^{-1}(E/Q))} (\operatorname{dis}(V, h), \sigma)_{\mathfrak{p}} \in Z^*.$$

(2) *If $\mathscr{P} = \bar{\mathscr{P}}$ over dyadic ramified of type I (meaning* $\operatorname{ord}_{\mathscr{P}} \Delta_{E/F} \equiv 1$ (mod 2), *see* [1, *p.* 7]) *then*

$$\partial(D, \mathscr{P})(V, h) = \operatorname{rank}(V, h) \in Z/2Z.$$

(3) *If $\mathscr{P} = \bar{\mathscr{P}}$ over dyadic ramified of type II (meaning* $\operatorname{ord}_{\mathscr{P}} \Delta_{E/F} \equiv 0$ (mod 2)) *then* $\partial(D, \mathscr{P})(V, h) = 0$.

(4) *If $\mathscr{P} = \bar{\mathscr{P}}$ over nondyadic ramified and if $\rho_{\mathscr{P}}$ is a local norm, then*

$$\partial(D, \mathscr{P})(V, h) = ((\operatorname{dis}(V, h), \sigma)_{\mathfrak{p}}, rk(V, h)).$$

(5) *If $\mathscr{P} = \bar{\mathscr{P}}$ over nondyadic ramified and if $\rho_{\mathscr{P}}$ is not a local norm*

$$\partial(D, \mathscr{P})(V, h) = ((-1)^{rk(V,h)} (\operatorname{dis}(V, h), \sigma)_{\mathfrak{p}}, rk(V, h)).$$

Proof. See [1] or [7]. ∎

Note that in case (5) when $\rho_{\mathscr{P}}$ is not a local norm, there is a correction term which may be obtained by multiplying by a rank 1 Witt class which corresponds under our identification of $W(F_{\mathfrak{p}})$ as $Z^* \times Z_2$ to the element $(-1, 1)$. We observe that on the fundamental ideal of even rank Witt classes $\partial(D, \mathscr{P})$ can be read independently of the choice of localizer.

Our discussion of the norm class of $\rho_{\mathscr{P}}$, namely $(\rho_{\mathscr{P}}, \sigma)_{\mathfrak{p}}$, is important in the distinction between cases (4) and (5).

EXAMPLE 4. Consider $E = Q(\sqrt{21}, \sqrt{-5})$ and let $F = Q(\sqrt{-5})$ be the fixed field of the Galois automorphism $\tau: \sqrt{21} \to -\sqrt{21}$ $\sqrt{-5} \to \sqrt{-5}$. Let $\theta = (1 + \sqrt{21} + 2\sqrt{21}\sqrt{-5})/2$. Under the involution τ we find that $\theta + \tau(\theta) = \theta + \bar{\theta} = 1$ and $\theta(\tau\theta) = \theta\bar{\theta} = 100 - 21\sqrt{-5}$. We may thus view E

as $E = F(\theta)$, where θ satisfies the monic irreducible $f(t) = t^2 - t + (100 - 21\sqrt{-5})$ over $F = Q(\sqrt{-5})$.

Remark. It is an interesting exercise to verify directly that θ is in the same square class as $\sqrt{21}$ in $Q(\sqrt{-5})$.

The involution τ on $E = F(\theta)$ arises from $\theta \to \bar{\theta} = 1 - \theta$. Thus the Hermitian group $H(E)$ will correspond to elements of the knot concordance group $C_S(Q)$ under $\mathrm{tr}_{E/Q}$.

Note that the primes 3 and 7 both split in $Q(\sqrt{-5})$, and they both ramify in $Q(\sqrt{21})$. It follows that we can write $3O(E) = \mathscr{P}_1^2 \mathscr{P}_2^2$ and similarly $7O(E) = \mathscr{P}_1'^2 \mathscr{P}_2'^2$. The prime 2 ramifies in $Q(\sqrt{-5})$ but is inert in $Q(\sqrt{-5})$. Note that $\mathrm{ord}_{\mathscr{P}} \Delta^{-1}(E/Q) \equiv 0 \pmod 2$ for $\mathscr{P}$ lying over 2. The prime 5 ramifies in $Q(\sqrt{-5})$ but splits in $Q(\sqrt{21})$. There are no signatures since $Q(\sqrt{-5})$ is totally imaginary.

The irreducible polynomial for θ over Q is $g(t) = t^4 - 2t^3 + 201t^2 - 200t + 12205$. Over $Z/3Z$, $g(t)$ factors as $g(t) = (t+1)^4$, and over $Z/7Z$, $g(t)$ factors as $g(t) = (t+3)^4$. Hence, by the description of prime ideals in the principal order $S = Z(\theta)$ given by Lagrange (see [6] or [7]) we find that there is only one prime ideal $\mathscr{M}_1$ in the nonmaximal order S lying over 3 and similarly there is only one prime ideal $\mathscr{M}_2$ lying over 7.

The boundary for the nonmaximal order S is computed by $\partial(S, \mathscr{M}) = \oplus_{\mathscr{P} \in T(\mathscr{M})} \mathrm{Tr}_{(D/\mathscr{P})/(S/\mathscr{M})} \circ \partial(D, \mathscr{P})$ (see [7, p. 147]). Of course $D/\mathscr{P} = S/\mathscr{M} = Z/3Z$ (or $Z/7Z$) at $\mathscr{M}_1$, $\mathscr{M}_2$, respectively, since at these ramified primes the inertial degree is equal to 1. Note that $T(\mathscr{M}_1) = \{\mathscr{P}_1, \mathscr{P}_2\}$ and $T(\mathscr{M}_2) = \{\mathscr{P}_1', \mathscr{P}_2'\}$ at the primes 3 and 7, respectively.

We have $E = F(\sqrt{21})$ where $\sigma = 21$ is clearly a local nonsquare in the local completions $\tilde{F}(\mathfrak{p}_1) = \tilde{F}(\mathfrak{p}_2) = \tilde{Q}(3)$ the 3-adics and $\tilde{F}(\mathfrak{p}_1') = \tilde{F}(\mathfrak{p}_2') = \tilde{Q}(7)$ the 7-adics. By Realization of Hilbert symbols, [5, p. 203] we can find $a \in F$ satisfying

$$
\begin{aligned}
(a, \sigma)_{\mathfrak{p}_1} &= +1, && \text{where } \mathscr{P}_1 \cap O(F) = \mathfrak{p}_1, \\
(a, \sigma)_{\mathfrak{p}_2} &= -1, && \text{where } \mathscr{P}_2 \cap O(f) = \mathfrak{p}_2, \\
(a, \sigma)_{\mathfrak{p}_1'} &= +1, && \text{where } \mathscr{P}_1' \cap O(F) = \mathfrak{p}_1', \\
(a, \sigma)_{\mathfrak{p}_2'} &= -1, && \text{where } \mathscr{P}_2' \cap O(F) = \mathfrak{p}_2', \\
(a, \sigma)_{\mathfrak{p}} &= +1 && \text{at all other primes } \mathfrak{p} \subset O(F).
\end{aligned}
$$

Finally consider the one-dimensional Hermitian form $(a) \in H(E)$. We claim: $\partial(S, \mathscr{M})(a) = 0$ for all $\mathscr{M} \subset S$. Clearly the local boundary vanishes at all primes except 2, 3, 5, and 7 by the choice of Hilbert symbols above together with the boundary computation Theorem 3.1. The prime 2 is inert in E/F, but $\mathrm{ord}_{\mathscr{P}} \Delta^{-1} \equiv 0 \pmod 2$ for $\mathscr{P}$ lying over 2 forces $\partial(D, \mathscr{P}) = 0$. The prime lying over 5 in $O(F)$ splits in $O(E)$ so that again boundary vanishes.

Consider the prime 3. The map $\mathrm{Tr}: D/\mathscr{P} \to S/\mathscr{M}$ is the identity as we have already observed. The canonical localizer ρ can be chosen to equal $(\frac{1}{2})(\frac{1}{3})$ by Theorem 2.7 since the ramification is tame. This is a local norm at both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ since $(6, 21)_3 = +1$. Hence by the boundary formula (4) we find $\partial(D, \mathscr{P}_1)(a) = (1, 1)$ and $\partial(D, \mathscr{P}_2)(a) = (-1, 1)$. Thus $\partial(S, \mathscr{M}_1)(a) = (1, 1) + (-1, 1) = (1, 0) = 0$. Similarly at 7 we find that $\rho = (\frac{1}{2})(\frac{1}{7})$ is a local norm at both $\mathfrak{p}_1'$ and $\mathfrak{p}_2'$ and the same computation as above yields $\partial(S, \mathscr{M}_2)(a) = 0$. Thus $\partial(S, \mathscr{M})(a) = 0$ for all $\mathscr{M}$, and by the boundary exact sequence there exists $(b) \in C_S(Z)$ which maps to (a). This produces the element of order 4 in $C_S(Z)$ since (a) clearly has order 4 in $H(E)$.

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